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Inflation and wavelets for the icosahedral Danzer tiling

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Abstract

The distribution of atoms in quasi-crystals lacks periodicity and displays point symmetry associated with non-crystallographic modules. Often it can be described by quasi-periodic tilings on $\mathbb{R}^3$ built from a finite number of prototiles. The modules and the canonical tilings of five-fold and icosahedral point symmetry admit inflation symmetry. In the simplest case of stone inflation, any prototile when scaled by the golden section number τ can be packed from unscaled prototiles. Observables supported on $\mathbb{R}^3$ for quasi-crystals require symmetry-adapted function spaces. We construct wavelet bases on $\mathbb{R}^3$ for the icosahedral Danzer tiling. The stone inflation of the four Danzer prototiles is given explicitly in terms of Euclidean group operations acting on $\mathbb{R}^3$. By acting with the unitary representations inverse to these operations on the characteristic functions of the prototiles, we recursively provide a full orthogonal wavelet basis of $\mathbb{R}^3$. It incorporates the icosahedral and inflation symmetry.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

We are interested in the construction of wavelets on quasi-periodic tilings (see also [8]). Wavelet bases are suited to analyse functions on quasi-crystals such as densities, electron and phonon states. Incorporation of specific tiling properties such as inflation will improve the efficiency of wavelet analysis. The simplest tilings are the ones with stone inflation. For these, any inflated tile can be packed from a finite set of prototiles. For functions with domains on the tiles this leads to very simple decompositions. The explicit construction of orthogonal wavelet bases for the Penrose–Robinson and the triangle tiling was given in [7]. We wish to extend this construction to three-dimensional icosahedral tilings, which are known to model stable classes of quasi-crystals [11]. As shown in [7], a basic step is to write the stone inflation of the tiling

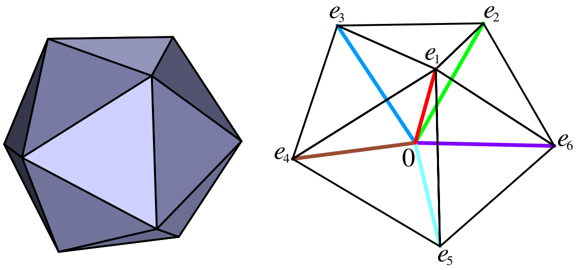

Figure 1. Icosahedron and axes $e_1, \ldots, e_6$.

in explicit form using operations from the Euclidean group $IO(3, \mathbb{R})$ in E^3. These Euclidean operations can then be lifted into linear unitary representations acting on the function spaces for wavelets. In what follows, we construct these Euclidean operations and the wavelets for the three-dimensional icosahedral Danzer tiling [5].

2. Icosahedral basis vectors, D_6 module, point groups

The extension of classical crystallography to quasi-crystals uses a module description as tools, see [14]. We shall need the primitive icosahedral P-module and the face-centred icosahedral F-module related to the root lattice D_6. These two modules are icosahedral projections from the primitive and the face-centred hypercubic lattice in six dimensions [11, 12].

The basis of the icosahedral primitive P-module in $E_\parallel$ (parallel space) is

$$\langle e_i, i = 1, \ldots, 6 \rangle. \tag{1}$$

These vectors are projections of six orthogonal unit vectors in E^6. Under icosahedral projection to $E_\parallel$ they point along six five-fold icosahedral axes and all have the standard length $|e_i| = (5) = \sqrt{1/2}$ (see figure 1). We shall use a bar overlining to denote a minus sign in front of a symbol.

For algebraic expressions w.r.t. a standard orthogonal basis, see [10]. In matrix form we have

$$\langle e_1, e_2, e_3, e_4, e_5, e_6 \rangle = \sqrt{1/2} \begin{bmatrix} 0 & s & \bar{s} & \bar{c} & 0 & c \\ s & c & c & 0 & \bar{s} & 0 \\ c & 0 & 0 & s & c & s \end{bmatrix} \tag{2}$$

where $s = \sin\beta$, $c = \cos\beta$, $\cot\beta = \tau = (1 + \sqrt{5})/2$, thus $s = (\tau + 2)^{-1/2}$ and $c = \tau s$. A basis of the icosahedral face-centred $2F = D_6$-module is

$$\langle e_1 - e_5, e_2 + e_4, e_3 + e_6, e_2 + e_3, e_3 + e_1, e_1 + e_2 \rangle. \tag{3}$$

The holes (vertices of Voronoï domains) of the lattice D_6 are of three types a, b, c with representatives, expressed by their coefficients in the basis equation (1),

$$a = \tfrac{1}{2}(111111) + D_6 \qquad c = \tfrac{1}{2}(11111\bar{1}) + D_6 \qquad b = \tfrac{1}{2}(200000) + D_6. \tag{4}$$

For the rotation axes of the icosahedral group, we introduce three sets of vectors along five-, two-, and three-fold axes. They are enumerated for each axis and indexed by the axis label:

$\boldsymbol{i_5}$ $\quad |i_5| = (5)$

$i_5 = e_i \qquad i = 1, \ldots, 6$

$\boldsymbol{j_2}$ $\quad |j_2| = (2)$

$$
\begin{array}{llll}
1_2 = e_1 - e_5 & 2_2 = e_1 - e_4 & 3_2 = e_1 - e_2 & 4_2 = e_1 - e_3 \\
5_2 = e_1 - e_6 & 6_2 = e_2 + e_4 & 7_2 = e_3 + e_5 & 8_2 = e_4 + e_6 \\
9_2 = e_5 + e_2 & 10_2 = e_6 + e_3 & 11_2 = e_4 - e_3 & 12_2 = e_5 - e_4 \\
13_2 = e_5 - e_6 & 14_2 = e_6 - e_2 & 15_2 = e_2 - e_3. &
\end{array} \tag{5}
$$

$\boldsymbol{l_3}$ $\quad |l_3| = (3)$

$$
\begin{array}{ll}
1_3 = (e_1 + e_2 + e_3 - e_4 + e_5 - e_6)/2 & 2_3 = (e_1 + e_3 + e_4 + e_6 - e_5 - e_2)/2 \\
3_3 = (e_1 + e_4 + e_5 + e_2 - e_6 - e_3)/2 & 4_3 = (e_1 + e_6 + e_5 + e_3 - e_4 - e_2)/2 \\
5_3 = (e_1 + e_6 + e_2 + e_4 - e_3 - e_5)/2 & 6_3 = (e_2 + e_3 - e_5 + e_6 + e_4 - e_1)/2 \\
7_3 = (e_3 + e_4 - e_6 + e_2 - e_1 + e_5)/2 & 8_3 = (e_4 + e_5 - e_2 + e_3 - e_1 + e_6)/2 \\
9_3 = (e_6 + e_5 - e_3 + e_2 + e_4 - e_1)/2 & 10_3 = (e_2 + e_6 - e_4 + e_3 + e_5 - e_1)/2.
\end{array} \tag{6}
$$

A standard choice for the length of the vectors along the three axes is [10, 11]

$$(5) = \sqrt{1/2} \qquad (2) = \sqrt{2/(\tau + 2)} \qquad (3) = \sqrt{3/2(\tau + 2)}. \tag{7}$$

The right-handed orthonormal triple of two-fold vectors $15_2, 1_2, 8_2$ is used for the algebraic description of all other vectors in [10]. Along the five-fold axes it yields the expressions of equation (2). For the stereographic representation of all axes, see [10].

We also need the axis vectors multiplied by τ but expressed in the basis $\langle e_1, \ldots, e_6 \rangle$. The rules for the construction in terms of these six vectors are given by inspection of the stereographic projection [10]. For each rotation axis, one can identify forward m-tuples of five-fold vectors which form an orbit under the rotation. These m-tuples may be used to write the inflation laws as follows:

$$
\begin{aligned}
&\tau i_5 = (\text{forward quintuple plus } i_5)/2 \\
&\tau^{-1} i_5 = (\text{forward quintuple minus } i_5)/2 \\
&j_2 = \text{wide forward pair} \\
&\tau j_2 = \text{narrow forward pair} \\
&l_3 = (\text{narrow forward triple} - \text{wide forward triple})/2 \\
&\tau l_3 = (\text{narrow forward triple} + \text{wide forward triple})/2.
\end{aligned}
$$

We shall express all operations of the icosahedral point groups as signed permutations of the vectors $e_i := i$ in cycle notation. They could also be expressed in terms of the icosahedral Coxeter group [9]. The generators of the icosahedral Coxeter group H_3 are, upon choosing the initial Coxeter cone with edge lines $3_3, 8_2, 1_5$,

$$\langle R_1 = (23)\,(46),\, R_2 = (45)(36),\, R_3 = (15)(2\overline{3})\rangle. \tag{8}$$

The reflections are generated by the Weyl vectors

$$\langle 15_2, \overline{12}_2, \overline{1}_2 \rangle. \tag{9}$$

The Coxeter relations of H_3 are

$$R_1^2 = R_2^2 = R_3^2 = e \qquad (R_1 R_2)^5 = (R_2 R_3)^3 = (R_3 R_1)^2 = e \tag{10}$$

where e means the identity.

Table 1. Standard position for Danzer prototiles: name, set of vertices, hole types, set of opposite outer normals and volume.

Prototile	Vertices	Hole type	Outer normals	Volume
A	$(0, 4_5, 8_2, 6_3)$	(c, a, c, b)	$(6_2, 15_2, \overline{9}_2, \overline{1}_2)$	$\tau^2/2$
A'	$(0, 6_5, 8_2, 6_3)$	(c, a, c, b)	$(10_2, \overline{15}_2, \overline{7}_2, \overline{1}_2)$	$\tau^2/2$
B	$(0, \tau 3_3, 8_2, \tau 1_5)$	(b, a, b, c)	$(4_2, 15_2, \overline{12}_2, \overline{1}_2)$	$\tau/2$
B'	$(0, \tau 4_3, 8_2, \tau 1_5)$	(b, a, b, c)	$(3_2, \overline{15}_2, \overline{13}_2, \overline{1}_2)$	$\tau/2$
C	$(0, \tau 3_3, \tau 8_2, 1_5)$	(a, b, a, c)	$(6_2, 15_2, \overline{12}_2, \overline{1}_2)$	$\tau/2$
C'	$(0, \tau 4_3, \tau 8_2, 1_5)$	(a, b, a, c)	$(10_2, \overline{15}_2, \overline{13}_2, \overline{1}_2)$	$\tau/2$
K	$(0, 4_5, 8_2/2, \tau^{-1} 1_5)$	$(a, c, -, b)$	$(8_2, 15_2, \overline{14}_2, \overline{1}_2)$	$1/4$
K'	$(0, 6_5, 8_2/2, \tau^{-1} 1_5)$	$(a, c, -, b)$	$(8_2, \overline{15}_2, \overline{11}_2, \overline{1}_2)$	$1/4$

3. The Danzer tiles

The Danzer tiling [4, 5] belongs to the $2F = D_6$ icosahedral module. It can be projected and derived by local rules from a canonical icosahedral tiling [12] $(\mathcal{T}, D_6)$. The latter tiling has six tiles which are projections of three-dimensional faces from the Voronoï domains of the D_6 lattice [10, 11]. This allows us to use the projection and window technique for the Danzer tiling. The Danzer tiling is also equivalent [6] to the Socolar–Steinhardt tiling [15].

The four tetrahedral prototiles are named [5] $\{A, B, C, K\}$. All edges of the prototiles run along five-, two- and three-fold icosahedral axes. All faces are perpendicular to two-fold axes. Each prototile has a mirror image which we denote for short by $\{A', B', C', K'\}$. In the Danzer tiling, four prototiles of the two types X, X' always appear glued to octahedra. We shall not use these octahedra but can see their appearance in the packings of inflated prototiles.

For an explicit algebraic description of the prototiles we choose for each prototile a reference vertex 0 and a standard orientation. For each prototile, we give in table 1 the vertex set in standard orientation and the outer normal vectors i_2 of the opposite face. For the mirror images of the four prototiles we shall use the prescription

$$X' = (23)\,(46)X. \tag{11}$$

The volumes[4] we write as fractions of the standard volume

$$V_0 = (\tau^{-2}/15)\sqrt{\frac{\tau+2}{2}}. \tag{12}$$

In table 1 we give in column 3 the hole type a, b, c of the four vertices according to equation (4).

Note that the third vertex of the prototiles K, K' is not a hole position. In the Danzer tiling it is completely covered by the octahedron formed from tiles K, K'. The positions of all defined tiles are demonstrated in figure 2.

4. Inflation of the Danzer prototiles

The Danzer tiling $\mathcal{T}$ admits a stone inflation [4, 5] and in fact was constructed by this stone inflation: any prototile $X \in \mathcal{T}$ when scaled by a factor τ can be packed face to face without gaps or overlaps from a set of translated and rotated prototiles. We call this the inflation of the prototiles. We denote by $X \to \tau X$ a linear scaling of the tile X with respect to its chosen

[4] Note for the reader: all tetrahedra have one vertex at the point [0,0,0] and the three others at points $[x_i, y_i, z_i]$, $i = 1, 2, 3$. Their volume can be calculated as $V = \frac{1}{6}|D|$, where $D = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$.

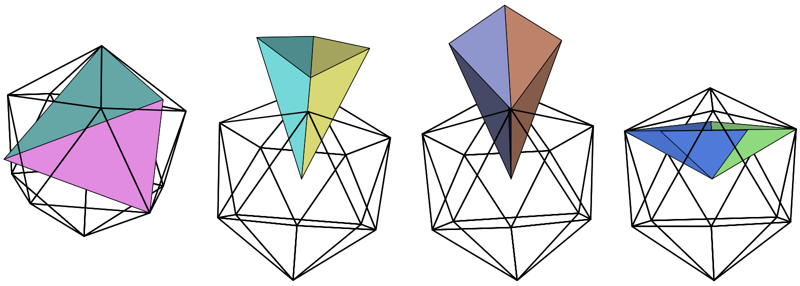

Figure 2. Tiles $A, A', B, B', C, C', K, K'$ and their positions in the icosahedron.

origin. Note that this origin in general is not the projection of a D_6 lattice point but rather the projection of a hole point (compare table 1). Inflation is a symmetry operation of the lattice D_6 when applied with respect to lattice points. With respect to hole points, inflation transforms classes of hole points into one another. When we apply inflation to the prototiles of the Danzer tiling, their vertices being projected as hole points are transformed into one another.

For the wavelet construction, we need explicit algebraic expressions of the inflation in terms of Euclidean group operations. We shall use the set of prototiles in the standard position according to table 1 along with their mirror images for the inflation. A prototile X_j that participates in the packing of the inflated tile τX is first rotated by a point group element g_j and then translated by a vector t_j. By writing $X_j \to (t_j, g_j)X_j$ we denote an Euclidean operation on the prototile X_j. Here $\gamma_j := (t_j, g_j)$ is an element of the Euclidean group $IO(3, \mathbb{R})$ acting on E^3, a rotation g_j followed by a translation t_j.

In the inflation of a fixed Danzer prototiles, a given prototile may participate in the packing more than once. In this case we need a second subscript to distinguish the corresponding occurrences. The inflation of the tile $X \in \{A, A', B, B', C, C', K, K'\}$ then takes the general form of a sum of Euclidean operators applied to the prototiles,

$$\tau X = \sum_j \left(\sum_{l_j} (t_{j,l_j}, g_{j,l_j}) X_j \right). \tag{13}$$

Inflation commutes with the full point group. Once we have obtained the inflation of τX, we obtain the inflation of $\tau X'$ from $\tau X' = \tau((23)\,(46))X = (23)\,(46)(\tau X)$ by the application of (23) (46) to the right-hand side of equation (13). On the right-hand side we can make the replacements $X_j = (23)\,(46)X'_j$, $X'_j = (23)\,(46)X_j$. If this is done, the elements of the Euclidean group appearing in equation (13) are conjugated with (23) (46) according to

$$(t_{j,l_j}, g_{j,l_j}) \to (0, (23)\,(46))(t_{j,l_j}, g_{j,l_j})(0, (23)\,(46)). \tag{14}$$

Inflation transforms the holes (a, c, b) into one another. From equation (4) and from the inflation rules for the vectors one finds in terms of classes the transformation rules

$$\tau a = c \qquad \tau c = b \qquad \tau b = a. \tag{15}$$

The vertices of the tiles belong to definite hole classes, and so the transformations, equation (15), apply to the vertices of the tiles upon τ-inflation.

We are now ready to give explicitly the expressions, equation (13), for the eight prototiles. For the inflation of the tiles $\{A', B', C', K'\}$ we apply equation (14).

In future constructions we shall apply inflation or its inverse to prototiles in positions and orientations other than the standard one. The corresponding decompositions under inflation can be obtained from equation (13) and table 2 by applying an element of the Euclidean group on both sides of equation (13) and using the multiplication rules for this group.

Table 2. Inflation of prototiles $\{A, A', B, B', C, C', K, K'\}$.

$\tau A=$		
(translation	, rotation	) prototile
$(\tau 8_2 = e_1 + e_5$	$, (1\overline{5}63\overline{4})$	$) B$
$+(\tau 8_2 = e_1 + e_5$	$, (2\overline{2})(5\overline{5})(1\overline{6})(34)$	$) B'$
$+(\tau 8_2 = e_1 + e_5$	$, (4\overline{4})(6\overline{6}))(1\overline{5})(23)$	$) B'$
$+(\tau 6_3 = (e_2 + e_3 - e_5 - e_4 - e_6 + e_1)/2$	$, (15\overline{26}3)$	$) C$
$+(\tau 6_3 = (e_2 + e_3 - e_5 - e_4 - e_6 + e_1)/2$	$, (15\overline{26}3)$	$) C'$
$+(\tau^{-1} 4_5 = (e_1 + e_3 - e_6 - e_2 + e_5 - e_4)/2$	$, (13\overline{62}5)$	$) K$
$+(\tau^{-1} 4_5 = (e_1 + e_3 - e_6 - e_2 + e_5 - e_4)/2$	$, (25364)$	$) K$
$+(\tau^{-1} 4_5 = (e_1 + e_3 - e_6 - e_2 + e_5 - e_4)/2$	$, (1\overline{42})(3\overline{5}\overline{6})$	$) K$
$+(\tau^{-1} 4_5 = (e_1 + e_3 - e_6 - e_2 + e_5 - e_4)/2$	$, (25364)$	$) K'$
$+(\tau^{-1} 4_5 = (e_1 + e_3 - e_6 - e_2 + e_5 - e_4)/2$	$, (13\overline{62}5)$	$) K'$
$+(\tau^{-1} 4_5 = (e_1 + e_3 - e_6 - e_2 + e_5 - e_4)/2$	$, (162)(35\overline{4})$	$) K'$
$\tau A' =$		
(translation	, rotation	) prototile
$(\tau 8_2 = e_1 + e_5$	$, (1\overline{5}42\overline{6})$	$) B'$
$+(\tau 8_2 = e_1 + e_5$	$, (3\overline{3})(5\overline{5})(1\overline{4})(26)$	$) B$
$+(\tau 8_2 = e_1 + e_5$	$, (6\overline{6})(4\overline{4}))(1\overline{5})(23)$	$) B$
$+(\tau 6_3 = (e_3 + e_2 - e_5 - e_6 - e_4 + e_1)/2$	$, (15\overline{34}2)$	$) C'$
$+(\tau 6_3 = (e_3 + e_2 - e_5 - e_6 - e_4 + e_1)/2$	$, (15\overline{34}2)$	$) C$
$+(\tau^{-1} 6_5 = (e_1 + e_2 - e_4 - e_3 + e_5 - e_6)/2$	$, (12\overline{43}5)$	$) K'$
$+(\tau^{-1} 6_5 = (e_1 + e_2 - e_4 - e_3 + e_5 - e_6)/2$	$, (35246)$	$) K'$
$+(\tau^{-1} 6_5 = (e_1 + e_2 - e_4 - e_3 + e_5 - e_6)/2$	$, (1\overline{63})(2\overline{5}\overline{4})$	$) K'$
$+(\tau^{-1} 6_5 = (e_1 + e_2 - e_4 - e_3 + e_5 - e_6)/2$	$, (35246)$	$) K$
$+(\tau^{-1} 6_5 = (e_1 + e_2 - e_4 - e_3 + e_5 - e_6)/2$	$, (12\overline{43}5)$	$) K$
$+(\tau^{-1} 6_5 = (e_1 + e_2 - e_4 - e_3 + e_5 - e_6)/2$	$, (143)(25\overline{6})$	$) K$
$\tau B =$		
(translation	, rotation	) prototile
$(\tau^2 1_5 = (e_2 + e_3 + e_4 + e_5 + e_6 + 3e_1)/2$	$, (1\overline{1})(6\overline{6})(2\overline{5})(3\overline{4})$	$) B$
$+(\tau^2 1_5 = (e_2 + e_3 + e_4 + e_5 + e_6 + 3e_1)/2$	$, (1\overline{2}\overline{5})(34\overline{6})$	$) B'$
$(0$	$, e$	$) C$
$+(\tau 8_2 = e_1 + e_5$	$, (132)(4\overline{5}6)$	$) K$
$+(\tau 8_2 = e_1 + e_5$	$, (1\overline{6}53\overline{2})$	$) K$
$+(\tau 8_2 = e_1 + e_5$	$, (1\overline{6}53\overline{2})$	$) K'$
$+(\tau 8_2 = e_1 + e_5$	$, (132)(4\overline{5}6)$	$) K'$
$\tau B' =$		
(translation	, rotation	) prototile
$(\tau^2 1_5 = (e_2 + e_3 + e_4 + e_5 + e_6 + 3e_1)/2$	$, (1\overline{1})(4\overline{4})(3\overline{5})(2\overline{6})$	$) B'$
$+(\tau^2 1_5 = (e_2 + e_3 + e_4 + e_5 + e_6 + 3e_1)/2$	$, (1\overline{3}\overline{5})(26\overline{4})$	$) B$
$+(0$	$, e$	$) C'$
$+(\tau 8_2 = e_1 + e_5$	$, (123)(6\overline{5}4)$	$) K'$
$+(\tau 8_2 = e_1 + e_5$	$, (1\overline{4}52\overline{3})$	$) K'$
$+(\tau 8_2 = e_1 + e_5$	$, (1\overline{4}52\overline{3})$	$) K$
$+(\tau 8_2 = e_1 + e_5$	$, (123)(6\overline{5}4)$	$) K$
$\tau C =$		
(translation	, rotation	) prototile
$(\tau 8_2 = e_1 + e_5$	$, e$	$) A$
$+(\tau^2 3_3 = (e_1 + e_4 + e_5)$	$, (3\overline{3})(5\overline{5})(1\overline{4})(26)$	$) C$
$+(\tau^2 3_3 = (e_1 + e_4 + e_5)$	$, (3\overline{3})(5\overline{5})(1\overline{4})(26)$	$) C'$
$+(1_5 = e_1$	$, (1\overline{24})(3\overline{6}\overline{5})$	$) K$
$+(1_5 = e_1$	$, (1\overline{24})(3\overline{6}\overline{5})$	$) K'$

Table 2. (Continued.)

$\boldsymbol{\tau C'} =$		
(translation	, rotation	) prototile
$(\tau 8_2 = e_1 + e_5$	, e	) A'
$+(\tau^2 4_3 = (e_1 + e_6 + e_5)$	, $(2\bar{2})(5\bar{5})(1\bar{6})(34)$	) C'
$+(\tau^2 4_3 = (e_1 + e_6 + e_5)$	, $(2\bar{2})(5\bar{5})(1\bar{6})(34)$	) C
$+(1_5 = e_1$	, $(1\bar{3}\bar{6})(2\bar{4}\bar{5})$	) K'
$+(1_5 = e_1$	, $(1\bar{3}\bar{6})(2\bar{4}\bar{5})$	) K
$\boldsymbol{\tau K} =$		
(translation	, rotation	) prototile
$(\tau 4_5 = (e_1 + e_3 - e_6 - e_2 + e_5 + e_4)/2$	, $(1\bar{4}\bar{2})(3\bar{5}\bar{6})$	) B
$+(1_5 = e_1$	, $(1\bar{2}\bar{4})(3\bar{6}\bar{5})$	) K
$\boldsymbol{\tau K'} =$		
(translation	, rotation	) prototile
$(\tau 6_5 = (e_1 + e_2 - e_4 - e_3 + e_5 + e_6)/2$	, $(1\bar{6}\bar{3})(2\bar{5}\bar{4})$	) B'
$+(1_5 = e_1$	, $(1\bar{3}\bar{6})(2\bar{4}\bar{5})$	) K'

In figure 3, we present all basic tiles $\{A, B, C, K\}$ in a bounding box, their surfaces cut by decomposition rules and edges of all composites. In figures 4, 5 and 6 we can see visualizations of decomposition rules from table 2 for the tiles A, B, C and K respectively. The sequence of pictures corresponds exactly with the sequence of tiles in table 2.

5. Haar wavelets for the Danzer tiling

We now construct an orthonormal Haar wavelet basis of $L^2(\mathbb{R}^3)$ associated with Danzer tiling $\mathcal{T}$ of $\mathbb{R}^3$. We recall that the Danzer tiling $\mathcal{T}$ of $\mathbb{R}^3$ is a set of compact tiles $(P_m)_{m\in\mathbb{N}}$ with the following properties:

(i) $\bigcup_{m\in\mathbb{N}} P_m = \mathbb{R}^3$,
(ii) $P_i \cap P_j \subset \mathbb{R}^2$ for $i \neq j$,
(iii) there is a finite set of prototiles $\mathcal{A} = \{T_i\}_{i=1}^8 = \{A, A', B, B', C, C', K, K'\}$ such that any tile P_m results from a translation and rotation (linear-affine transformation) of an element of $\mathcal{A}$, (this means that $P_m = \gamma_k T_k$ where $1 \leqslant k \leqslant 8$ and $\gamma_k \in \Gamma_k$. The set Γ_k is the set of all admissible linear-affine transformations of a particular tile T_k.)
(iv) for all $m \in \mathbb{N}$, τP_m is the union of finitely many P_i, see table 2,
(v) $\sigma\mathcal{T} \subset \mathcal{T}$, where σ is the stone-inflation transformation.

Let us define a sequence of spaces $(V_j(\mathcal{T}))_{j\in\mathbb{Z}}$ such that each space $V_j(\mathcal{T})$ is a closed subspace of $L^2(\mathbb{R}^3)$ of functions which are constant on all tiles $\sigma^{-j} P_m$, $m \in \mathbb{N}$. We denote by $\gamma_i = (t_i, g_i)$ a translation-rotation in the Euclidean group $IO(3, \mathbb{R})$ which brings a tile T_i to one of its congruent companions in the tiling, and by Γ_i the set of these operations.

Proposition 1. *The sequence $(V_j(\mathcal{T}))_{j\in\mathbb{Z}}$ is a σ-multiresolution analysis of $L^2(\mathbb{R}^3)$. Thus it obeys the following properties:*

(i) For all $j \in \mathbb{Z}$, $V_{j-1}(\mathcal{T}) \subset V_j(\mathcal{T})$,
(ii) $\bigcup_{j\in\mathbb{Z}} V_j(\mathcal{T})$ is dense in $L^2(\mathbb{R}^3)$,
(iii) $\bigcap_{j\in\mathbb{Z}} V_j(\mathcal{T}) = \{0\}$,
(iv) for all $j \in \mathbb{Z}$, $\mathbf{x} \in \mathbb{R}^3$, $f(\mathbf{x}) \in V_j(\mathcal{T}) \Longleftrightarrow f(\sigma^{-j}\mathbf{x}) \in V_0(\mathcal{T})$,
(v) there exist eight scaling functions, $\phi_1(\mathbf{x}), \phi_2(\mathbf{x}), \ldots, \phi_8(\mathbf{x})$ such that all their admissible linear-affine transformations $\{\phi_i(\gamma_i^{-1}\cdot\mathbf{x})\}_{1\leqslant i\leqslant 8, \gamma_i\in\Gamma_i}$ form an orthonormal basis in $V_0(\mathcal{T})$.

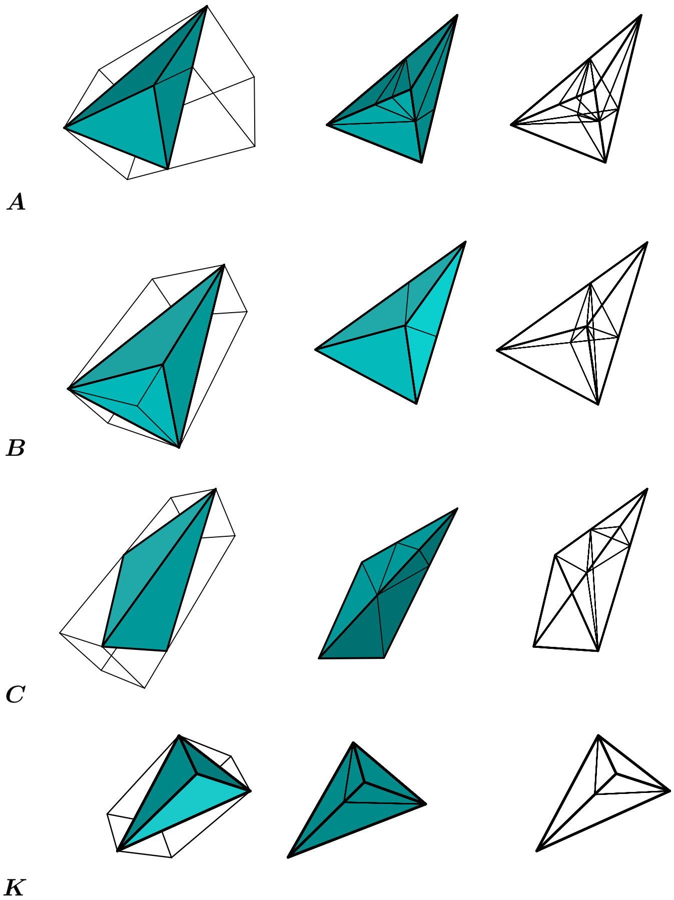

Figure 3. The tiles A, B, C and K.

Proof.

(i) This inclusion results from the stone-inflation property of tiling $\mathcal{T}$, $\sigma\mathcal{T} \subset \mathcal{T}$.

(ii) This is true through the fact that every continuous function f with a compact support on $\mathbb{R}^3$ can be written as the uniform limit of the sequence $(f_j)_{j\geqslant 0}$ such that

$$f_j(\mathbf{x}) = \sum_{m\in\mathbb{N}} f(\mathbf{x}_{j,m})\chi_{\sigma^{-j}P_m}(\mathbf{x})$$

where $\mathbf{x}_{j,m} \in \sigma^{-j}P_m$ and $\chi_{\sigma^{-j}P_m}(\mathbf{x})$ is the characteristic function of the tile $\sigma^{-j}P_m$.

(iii) By construction, it is clear that only the function $f(\mathbf{x}) = 0$ is included in all spaces $V_j(\mathcal{T})$.

(iv) Let us choose $f(\mathbf{x}) \in V_j(\mathcal{T})$. Then we have

$$f(\mathbf{x}) = \sum_{m\in\mathbb{N}} c_m\chi_{\sigma^{-j}P_m}(\mathbf{x}) \qquad \text{with} \qquad \sum_{m\in\mathbb{N}} |c_m|^2 < \infty$$

and this is equivalent to

$$f(\sigma^{-j}\mathbf{x}) = \sum_{m\in\mathbb{N}} c_m\chi_{P_m}(\mathbf{x}).$$

Thus $f(\sigma^{-j}\mathbf{x}) \in V_0(\mathcal{T})$.

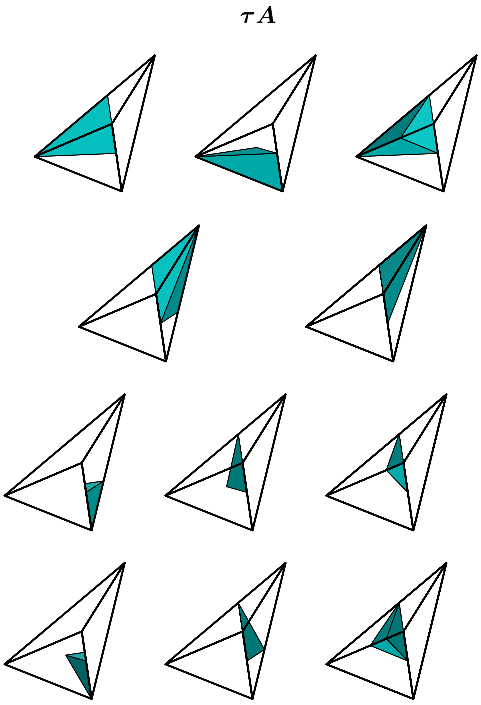

Figure 4. Decomposition of tile τA resulting from table 2.

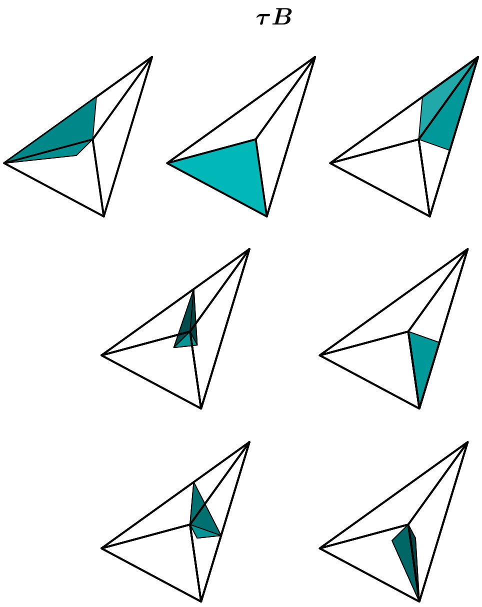

Figure 5. Decomposition of tile τB resulting from table 2.

(v) We have eight scaling functions $\phi_1(\mathbf{x}), \phi_2(\mathbf{x}), \ldots, \phi_8(\mathbf{x})$ which are the characteristic functions of the corresponding prototiles T_i, $i = 1, \ldots, 8$, normalized by division with

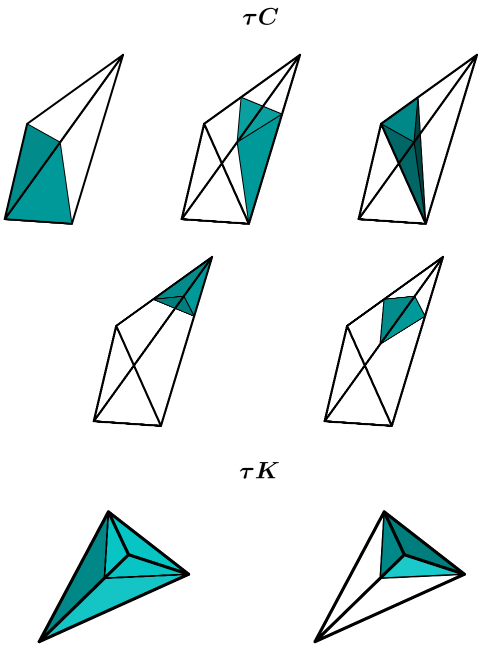

Figure 6. Decomposition of tiles τC and τK resulting from table 2.

the square root of the volume of the tile, e.g., for a tile $X \in \mathcal{A}$

$$\phi_X(\mathbf{x}) = \chi_X(\mathbf{x})/\sqrt{|X|}$$

where $|X|$ means the volume of the tile X. Due to the normalization, we see that all admissible linear-affine transformations $\gamma_i \in \Gamma_i$ of eight functions $\{\phi_i\}_{i=1}^{8}$ form an orthonormal basis of $V_0(\mathcal{T})$. □

The construction of the Haar wavelets basis for $L^2(\mathbb{R}^3)$ associated with stone-inflation Danzer tiling $\mathcal{T}$ will be done in the following way [1, 2, 7]. We recall that we have a sequence of spaces $V_j(\mathcal{T})$ forming a σ-multiresolution analysis of $L^2(\mathbb{R}^3)$ and eight prototiles in tiling $\mathcal{T}$, $\mathcal{A} = \{T_i\}_{i=1}^{8} = \{A, A', B, B', C, C', K, K'\}$. For each prototile T_i, $1 \leqslant i \leqslant 8$ there are finitely many tiles P_{i_j}, such that:

$$\sigma T_i = P_{i_1} \cup \cdots \cup P_{i_{k_i}} \Longleftrightarrow T_i = \sigma^{-1} P_{i_1} \cup \cdots \cup \sigma^{-1} P_{i_{k_i}}.$$

Now we denote by $V_{0,P_m}(\mathcal{T})$ a subspace of $V_0(\mathcal{T})$ of functions which are zero (almost everywhere) outside the tile P_m. Consequently, we define $V_{j,P_m}(\mathcal{T})$ as a subspace of $V_j(\mathcal{T})$ of functions equal to zero (almost everywhere) outside the tile P_m. Therefore, we obtain orthogonal decompositions:

$$V_0(\mathcal{T}) = \bigoplus_{m\in\mathbb{N}}{}_{\perp} V_{0,P_m}(\mathcal{T}) \tag{16}$$

$$V_j(\mathcal{T}) = \bigoplus_{m\in\mathbb{N}}{}_{\perp} V_{j,P_m}(\mathcal{T}) \tag{17}$$

and the inclusion

$$V_{0,P_m}(\mathcal{T}) \subset V_{1,P_m}(\mathcal{T}) \subset \cdots \subset V_{j,P_m}(\mathcal{T}) \subset \cdots. \tag{18}$$

Then the wavelet space $W_0(\mathcal{T})$ is an orthogonal complement of $V_0(\mathcal{T})$ in $V_1(\mathcal{T})$,

$$V_1(\mathcal{T}) = V_0(\mathcal{T}) \oplus_\perp W_0(\mathcal{T}). \tag{19}$$

More generally

$$V_{j+1}(\mathcal{T}) = V_j(\mathcal{T}) \oplus_\perp W_j(\mathcal{T}).$$

We can also define an orthogonal complement of $V_{j,P_m}(\mathcal{T})$ in $V_{j+1,P_m}(\mathcal{T})$ as

$$V_{j+1,P_m}(\mathcal{T}) = V_{j,P_m}(\mathcal{T}) \oplus_\perp W_{j,P_m}(\mathcal{T}) \tag{20}$$

thus we have

$$W_j(\mathcal{T}) = \bigoplus_{m\in\mathbb{Z}}{}_{\perp} W_{j,P_m}(\mathcal{T}).$$

And consequently we have

$$L^2(\mathbb{R}^3) = \bigoplus_{j\in\mathbb{Z}}{}_{\perp} W_j(\mathcal{T}).$$

Thus, the construction of $W_0(\mathcal{T})$ is equivalent to the construction of all $W_{0,P_m}(\mathcal{T})$. Since any tile P_m can be written as

$$P_m = \gamma_i T_i = \gamma_i\left(\sigma^{-1}P_{i_1} \cup \cdots \cup \sigma^{-1}P_{i_{k_i}}\right) \qquad \gamma_i \in \Gamma_i$$

it is sufficient to find wavelets for our prototiles T_i, $i = 1, \ldots, 8$, and the whole basis of $W_0(\mathcal{T})$ will be formed by all admissible linear-affine transformations of these 'protowavelets'. There results the following proposition:

Proposition 2. *For every prototile T_i, $i = 1, \ldots, 8$, given also by*

$$T_i = \sigma^{-1}P_{i_1} \cup \cdots \cup \sigma^{-1}P_{i_{k_i}}$$

we have $k_i - 1$ orthonormal wavelets $\psi_{1,i}(\mathbf{x}), \ldots, \psi_{k_i-1,i}(\mathbf{x})$.

Proof. Let us denote by $V_{0,T_i}(\mathcal{T})$ the space of functions constant on tile T_i and equal to zero otherwise. We then denote by $V_{1,T_i}(\mathcal{T})$ the space of functions constant on tiles $\sigma^{-1}P_{i_1}, \ldots, \sigma^{-1}P_{i_{k_i}}$ and otherwise equal to zero. The space of wavelets corresponding to the tile T_i is found as the orthogonal complement $V_{0,T_i}(\mathcal{T})$ in $V_{1,T_i}(\mathcal{T})$, i.e.

$$V_{1,T_i}(\mathcal{T}) = V_{0,T_i}(\mathcal{T}) \oplus_\perp W_{0,T_i}(\mathcal{T}).$$

The space $V_{1,T_i}(\mathcal{T})$ is a linear span of k_i linearly independent functions $\left\{\chi_{\sigma^{-1}P_{i_1}}(\mathbf{x}), \ldots, \chi_{\sigma^{-1}P_{i_{k_i}}}(\mathbf{x})\right\}$, the space $V_{0,T_i}(\mathcal{T})$ consists of only one function $\chi_{T_i}(\mathbf{x})$ and its multiples. Thus, the space $W_{0,T_i}(\mathcal{T})$ has to be $k_i - 1$ dimensional. We get an orthonormal basis of $W_{0,T_i}(\mathcal{T})$ by an application of the following lemma: □

Lemma 1. *Given n orthogonal characteristic functions $\chi_1, \ldots, \chi_n$ on a unitary space with the scalar products*

$$\langle \chi_i, \chi_j \rangle = \delta_{ij}\|\chi_i\|^2 = \delta_{ij}(f_i)^2 \tag{21}$$

a new orthonormal set $\varphi_1, \ldots, \varphi_n$ can be constructed from them as

$$\begin{aligned} \varphi_1 &:= \frac{f_2}{F_2 f_1}\chi_1 - \frac{f_1}{F_2 f_2}\chi_2 \\ \varphi_2 &:= \frac{f_3}{F_3 F_2}(\chi_1 + \chi_2) - \frac{F_2}{F_3 f_3}\chi_3 \\ &\ \ \vdots \end{aligned} \tag{22}$$

$$\varphi_{n-1} := \frac{f_n}{F_n F_{n-1}} \sum_{i=1}^{n-1} \chi_i - \frac{F_{n-1}}{F_n f_n} \chi_n$$
$$\varphi_n := \frac{1}{F_n} \sum_{i=1}^{n} \chi_i \tag{23}$$

where F_m is defined by

$$F_m := \left[\sum_{i=1}^{m} (f_i)^2 \right]^{1/2}. \tag{24}$$

This lemma gives a simple procedure of how to obtain from an orthogonal basis of $V_{1,X}(\mathcal{T})$, $X \in \mathcal{A}$, formed by n characteristic functions living on corresponding subtiles, an orthonormal basis of $W_{0,X}(\mathcal{T})$. The last function φ_n is proportional to the sum of the characteristic functions of all subtiles, thus it is the normalized characteristic function of tile X, $\varphi_n = \chi_X(\mathbf{x})/\sqrt{|X|}$. The orthonormality condition for the functions constructed assures that the remaining functions $\varphi_1, \ldots, \varphi_{n-1}$ form an orthonormal basis of $W_{0,X}(\mathcal{T})$. We note that one can interpret figures 4–6 as a visualization of the basis (created by characteristic functions of corresponding subtiles) of $V_{1,X}(\mathcal{T})$, $X \in \mathcal{A}$.

Proposition 2 also says how many different orthonormal wavelets we need for a given prototile. For the prototiles $\{T_i\}_{i=1}^{8} = \{A, A', B, B', C, C', K, K'\}$, we need $\{k_i - 1\}_{i=1}^{8} = \{10, 10, 6, 6, 4, 4, 1, 1\}$ wavelets which can be found by using the previous lemma. The orthonormal Haar wavelet basis of $L^2(\mathbb{R}^3)$ adapted to the Danzer tiling $\mathcal{T}$ is then given by

$$\bigcup_{j\in\mathbb{Z}} \bigcup_{i=1}^{8} \bigcup_{\gamma_i\in\Gamma_i} \left\{ \tau^{j\frac{3}{2}} \psi_{l,i}\left(\gamma_i^{-1}\sigma^j \cdot \mathbf{x}\right) \right\}_{l=1}^{k_i-1}.$$

The construction of Haar wavelets for any stone-inflation n-dimensional tiling can be easily derived from the present example.

6. Characteristic functions and boundaries of the Danzer tiling

The wavelet basis was constructed from the characteristic functions $\chi_{X_i}(x)$ on the tiles $X_i \in \mathcal{T}$. For two adjacent tiles X_i, X_j, the additive property

$$\chi_{\{X_i \cup X_j\}}(x) = \chi_{X_i}(x) + \chi_{X_j}(x) \tag{25}$$

holds true for all strictly exterior points $x \in \mathrm{Ext}(X_i \cup X_j)$ and for the interior points $x \in \mathrm{Int}(X_i)$, $x \in \mathrm{Int}(X_j)$, but fails for points on the boundary $x \in (X_i \cap X_j)$ (which form a set of measure zero). To have the additive property, equation (25), for all $x \in \mathbb{R}^3$, and to represent functions with non-zero values on points of boundaries, appropriate values of the characteristic functions must be specified on all the boundaries of the tiling.

Suppose it were possible to link any boundary in the Danzer tiling $\mathcal{T}$ in a unique local way to a single tile X_i. Then it would be possible to extend the set of points with value $\{x \mid \chi_{X_i}(x) = 1\}$ from the interior $x \in \mathrm{Int}(X_i)$ to all the points on all boundaries linked to it, and the additive property equation (25) would hold true on $\mathbb{R}^3$.

We shall give here one possible solution for this local linkage of boundaries to tiles. We (I) interpret $\mathcal{T}$ as a locally finite simplicial complex, (II) associate sets of unit vectors with all simplices and (III) select from the set of simplices bounded by any fixed simplex a unique simplex chain of increasing dimension which ends on a unique 3-simplex.

(I) The Danzer tiling $\mathcal{T}$ consists of 3-simplices (tetrahedral tiles) bounded by subsimplices (triangles, edges, vertices) of dimensions $p = 2, 1, 0$. Any intersection of simplex boundaries is another simplex boundary of $\mathcal{T}$, and any pair of distinct simplices have disjoint interiors. Therefore, $\mathcal{T}$ is a simplicial complex as defined in algebraic topology [13, p 7]. Moreover, since any subsimplex belongs to only finitely many (sub-)simplices, the simplicial complex $\mathcal{T}$ is locally finite [13, p 11].

(II) We denote the simplices of $\mathcal{T}$ of dimension $p = 2, 1, 0$ by $\sigma(p)$, distinguish them if necessary by greek indices and associate outer unit vectors $n(\sigma(p))$ with them. The unit vectors allow us to classify the bounding property $\sigma(p) \subset \sigma(p+1)$, $p = 2, 1, 0$.

It will be necessary to distinguish unit vectors of opposite direction: at all points of $\mathbb{R}^3$ we choose the same set of polar coordinates θ, ϕ on the unit sphere S^2 and define the upper half-sphere by $S^{2,+} = \{(\theta, \phi) \mid \{0 \leqslant \theta < \pi/2, 0 \leqslant \phi < 2\pi\} \cup \{\theta = \pi/2, 0 \leqslant \phi < \pi\}\}$. The unit vectors $n(\sigma(2)) \in S^{2,+}$ we order by the relation $n_{\sigma(\alpha)} \prec n_{\sigma(\beta)} \leftrightarrow \theta_\alpha < \theta_\beta$ or $\theta_\alpha = \theta_\beta, \phi_\alpha < \phi_\beta$.

$p = 2$: To any 2-simplex boundary $\sigma(2)$ of a 3-simplex $\sigma(3) \in \mathcal{T}$, we associate four unit vectors $n(\sigma(2))$ normal to the 2-simplex and pointing outward w.r.t. $\sigma(3)$. These vectors for tiles in standard position up to normalization were given in table 1. Clearly for $\sigma(2) \in \mathcal{T}$, any such unit vector $n(\sigma(2))$ points along one of the 15 two-fold axes of the icosahedral group. To a given 2-simplex $\sigma(2)$ are associated two normal outer vectors $n(\sigma(2, \mu)), n(\sigma(2, \mu'))$ of opposite direction, corresponding to two 3-simplices $\sigma(3, \mu), \sigma(3, \mu')$ bounded by $\sigma(2)$.

$p = 1$: To any 1-simplex $\sigma(1)$ bounding a 2-simplex $\sigma(2)$ we associate a unit vector $n(\sigma(1))$ perpendicular to the 1-simplex, in the plane embedding $\sigma(2)$, and pointing outward from $\sigma(2)$. The 1-simplices of $\mathcal{T}$, being perpendicular to at least two two-fold axes, all are on two-fold, three-fold or five-fold axis lines of the icosahedral group. The full set of unit vectors $n(\sigma(1))$ associated with a fixed $\sigma(1)$ is distributed into all the planes of 2-simplices bounded by $\sigma(1)$. At most two vectors $n(\sigma(1, \nu)), n(\sigma(1, \nu'))$ can be in the same plane containing $\sigma(1)$. If this happens, $\sigma(1)$ bounds two different 2-simplices $\sigma(2, \mu), \sigma(2, \mu')$ in this plane. This enforces $n(\sigma(1, \nu)) = -n(\sigma(1, \nu'))$.

$p = 0$: To a 0-simplex $\sigma(0)$ bounding a 1-simplex $\sigma(1)$ we associate a vector $n(\sigma(0))$ within the line given by $\sigma(1)$ and pointing outward from $\sigma(1)$ at the end point $\sigma(0)$. The vectors $n(\sigma(0, \nu))$ point along the lines of all 1-simplices bounded by $\sigma(0)$. A fixed one of them $\sigma(1, \rho)$ bounds itself a set of 2-simplices $\sigma(2, \mu)$. A fixed $\sigma(2, \mu)$ bounded by $\sigma(0)$ fixes a plane perpendicular to $n(\sigma(2, \mu))$ to which all vectors $n(\sigma(1, \rho))$ with $\sigma(0) \in \sigma(1, \rho) \in \sigma(2, \mu)$ must belong. For fixed $\sigma(0)$ and fixed $n(\sigma(2, \mu))$, there can be at most two different 1-simplices $\sigma(1, \rho), \sigma(1, \rho')$ such that $n(\sigma(1, \rho)) = n(\sigma(1, \rho'))$. In this case $\sigma(1, \rho)$ and $\sigma(1, \rho')$ end at $\sigma(0)$ from opposite sides and can be distinguished by the vectors $n(\sigma(0), \rho) = -n(\sigma(0), \rho')$.

The unit vectors for the 2-, 1- and 0-simplices classify all the simplices bounded by $\sigma(0)$.

(III) With the help of the unit vectors $n(\sigma(p))$ we now link any $\sigma(p)$ to a unique chain $\sigma(p) \subset \sigma(p+1) \subset \cdots \subset \sigma(3)$, $p = 0, 1, 2$ of bounding simplices.

$p = 2$: A fixed 2-simplex $\sigma(2)$ bounds two 3-simplices. We link it to the 3-simplex whose outer normal unit vector fulfils $n(\sigma(2)) \in S^{2,+}$.

$p = 1$: A fixed 1-simplex $\sigma(1)$ bounds a finite set $\sigma(2, \mu)$ of 2-simplices. Choosing their normal unit vectors $n(\sigma(2, \mu)) \in S^{2,+}$ each one is linked to a 3-simplex. We determine from this set $n(\sigma(2,\mu_0))$ as the minimal unit vector w.r.t. the order $\prec$. If there is a single 2-simplex $\sigma(2,\mu_0)$ in the subset with $n(\sigma(2)) = n(\sigma(2,\mu_0))$ we link $\sigma(1)$ to it. Otherwise, there are two different 2-simplices $\sigma(2,\mu), \sigma(2,\mu')$ with $n(\sigma(2,\mu)) = n(\sigma(2,\mu')) = n(\sigma(2,\mu_0))$, and $\sigma(1)$ must bound this pair of 2-simplices $\sigma(2,\mu), \sigma(2,\mu')$ within the same plane. Then the two vectors $n(\sigma(1),\mu), n(\sigma(1),\mu')$ pointing outward from $\sigma(2,\mu), \sigma(2,\mu')$ must have opposite direction. We select the 2-simplex with $n(\sigma(1),\mu) \in S^{2,+}$, link $\sigma(1)$ to it and continue the chain to a unique $\sigma(3)$.

$p = 0$: A fixed 0-simplex $\sigma(0)$ bounds a finite set of 1-boundaries, each one of them linked already to a unique 2-boundary and 3-simplex. From this set of 1-boundaries we select the subset linked to 2-boundaries with the minimal unit vector $n(\sigma(2,\mu_0)) \in S^{2,+}$ w.r.t. the order $\prec$. If there is a single $\sigma(1)$ in this subset, we link $\sigma(0)$ to it and continue the chain. If several 1-simplices $\sigma(1,\nu)$ form the subset associated with $n(\sigma(2,\mu_0)) \in S^{2,+}$, all their unit vectors $n(\sigma(1,\nu))$ must be in the plane perpendicular to $n(\sigma(2,\mu_0))$. We order the unit vectors for this second subset by an angle in this plane and choose a minimal one $n(\sigma(1,\nu_0))$. If $\sigma(1,\rho_0)$ is uniquely determined by this vector, we link $\sigma(0)$ to it. If not, there are two 1-simplices $\sigma(1,\rho_0), \sigma(1,\rho_0')$ distinguished by vectors $\sigma(0,\rho_0), \sigma(0,\rho_0')$ of opposite direction. We link $\sigma(0)$ to $\sigma(1,\rho_0)$ such that $\sigma(0,\rho_0) \in S^{2,+}$ and continue the chain to a unique $\sigma(3)$.

Any boundary $\sigma(p)$ of the simplicial complex $\mathcal{T}$ is linked by these constructions to a unique chain of bounding simplices $\sigma(p) \subset \sigma(p+1) \subset \cdots \subset \sigma(3)$, $p = 0, 1, 2$, ending with a unique 3-simplex or tile of $\mathcal{T}$. With this result we get for characteristic functions on $\mathcal{T}$ the following lemma:

Lemma 2. *Consider any tile $X_i \in \mathcal{T}$. Construct its characteristic function $\chi_{X_i}(x)$ on $\mathbb{R}^3$ by assigning the value $\chi_{X_i}(x) = 1$ to any interior point, and to any point x of a simplex chain ending on X_i, and put $\chi_{X_i}(x) = 0$ otherwise. Do this construction on all the tiles. Then the characteristic functions on $\mathcal{T}$ have the additive property, equation (25), on the points of all p-simplices in $X_i \cap X_j$ linked to X_i or X_j. To assure an additive property on other p-simplices in $X_i \cap X_j$, it will be necessary to consider the tiles linked to them and their characteristic functions.*

Without going into details we assert: when the characteristic functions for wavelet bases are extended by lemma 2, they allow for the expansion of functions which take non-zero values only on subsimplices of $\mathcal{T}$.

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